L26 The logarithm Function, part I (Conti.) (續.對數函數)
7.3 The logarithm function, part II
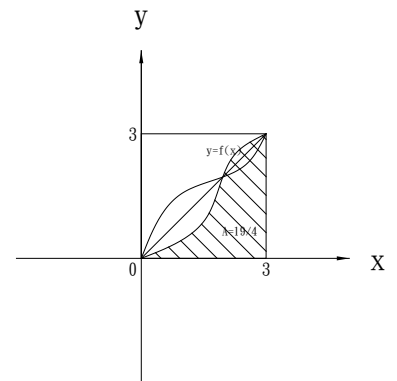7.4 The exponential function (指數函數)

Let f:[0,3]→[0,3] be a cont. one-to-one function with f(0)=0 and f(3)=3. If

$\int_0^3 f(x)dx = \dfrac{19}{4}$, compute $\int_0^3 f^{-1}(x)dx = ?$ (Hint:畫圖)

如果一個函數一對一和連續，則函數不是遞增就遞減。
By the way 一般題目不給提示，除非不是從所求想起。
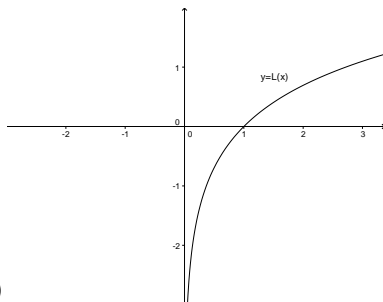
$\int_0^3 f^{-1}(x)dx = 3\times 3 - \dfrac{19}{4} = \dfrac{17}{4}$

Rmk:We will show that 2<e<3.

conclusion:

① lnx=$\int_1^x \dfrac{1}{t}dt$, ∀ x∈(0,∞).

②

③ $\int_1^e \dfrac{1}{t}dt$ =1 or lne=$\log_e$ e=1   (圖形上表示 1/x，1 積到e的面積為 1)

④ lne^p/q=p/q.

Ex:P346(22.23.24.25)

§ 7.3 The logarithm function, part 2

Prop:

d/dxln|x|=1/x, ∀ x≠0.   加絕對可以拿掉性質符號。

pf:

(i) For x>0, d/dx ln|x|= d/dx lnx=1/x.

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(ii) For x<0, d/dx ln|x|= d/dx ln(-x)=-1/x·(-1)=1/x.

Thm: $\int \dfrac{dx}{x} = \ln|x| + C$,for $x \neq 0$ 因為 x 可能正可能負，所以加上絕對值。

Thm: $\int \dfrac{g'(x)}{g(x)} = \ln|g(x)| + C$

pf:Let u=g(x), then du=g'(x)dx

原式 $\int \dfrac{du}{u} = \ln|u| + C = \ln|g(x)| + C$

Thm:六個三角函數都可以積，連續函數一定可積。

① $\int \tan x dx = \ln|\sec x| + C$

② $\int \cot x dx = -\ln|\csc x| + C$

③ $\int \sec x dx = \ln|\sec x + \tan x| + C$

④ $\int \csc x dx = -\ln|\csc x + \cot x| + C$

pf:

① $\int \tan x dx$

Let u=cosx, then du=-sinxdx

$= \int \dfrac{\sin x}{\cos x} dx = -\ln|\cos| + C = \ln|\sec x| + C$

③ $\int \sec x dx$

$= \int \sec x dx \times \dfrac{\sec x + \tan x}{\sec x + \tan x} = \int \dfrac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx = \ln|\sec x + \tan x| + C$

By the way~[f(x)+f¹(x)]/2=x 這是不對的。圖 L26-2

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e.g

① $\int \dfrac{x^2}{1-4x^3}dx = -\dfrac{1}{12}\ln\left|1-4x^3\right|+C$

② $\int_1^e \dfrac{\ln x}{x}dx = \dfrac{1}{2}\ln^2 x\Big|_1^e = \dfrac{1}{2}(1-0)=\dfrac{1}{2}$

③ $\int_1^3 \dfrac{6x^2-2}{x^3-x+1}dx = 2\int_*^{**}\dfrac{du}{u} = 2\ln\left|x^3-x+1\right|\Big|_1^3 = 2(\ln 25-0)=2\ln 25$

④ $(\ln\dfrac{x+1}{\sqrt{x-2}})' = \dfrac{\sqrt{x-2}}{x+1}\cdot\dfrac{\sqrt{x-2}-(x+1)\cdot\dfrac{1}{2}\cdot\dfrac{1}{\sqrt{x-2}}}{x-2} = \ln\left|\dfrac{1}{x+1}\right|-\dfrac{1}{2}\ln\left|\dfrac{1}{x-2}\right|+C$

Ex:P354(8.12.14.22.23.24.27.31.32.33.34)

§ 7.4 The exponential function

∵ $\ln:\mathbb{R}^+\to\mathbb{R}$ is one-to-one.

∴ $\ln^{-1}:\mathbb{R}\to\mathbb{R}^+$ exists.

Q:ln 它的值域是多少？A:負無限到正無限大。

Def: Let $E(x)=\ln^{-1}x$, then $E:\mathbb{R}\to\mathbb{R}^+$.

Prop:
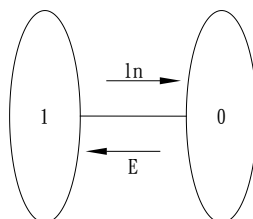
① $E(0)=1$ 從lnx的圖形去想

② Let $p/q\in\mathbb{Q}$, then $E(p/q)=e^{p/q}$

③ $E(x+y)=E(x)E(y)$, $\forall$ $x,y\in\mathbb{R}$ 等號還沒有成立，要證等號成立。

④ $E(-x)=1/E(x)$, $\forall$ $x\in\mathbb{R}$

⑤ $E(x)>0$, $\forall$ $x\in\mathbb{R}$

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pf:

① ∵ ln1=0 ∴ E(0)=1

② ∵ lne^p/q=p/q ∴E(p/q)=e^p/q

③ lnE(x+y)=x+y,lnE(x)=x,lnE(y)=y

  ⇒ lnE(x+y)= lnE(x)+ lnE(y)=ln[E(x)E(y)]

  ∵ ln is one-to-one ∴ E(x+y)=E(x)E(y) 取值一樣，自變數一樣

④ ∵ E(x)·E(-x)=by③ E(x-x)=E(0)=1 ∴ E(-x)=1/E(x)

⑤ ∵ ln:$\mathbb{R}^+$→$\mathbb{R}$ is one-to-one ∴ ln^{-1}:$\mathbb{R}$→$\mathbb{R}^+$ 因爲它的值於$\mathbb{R}^+$

Rmk:From①,③,④,⑤，可知E(x)滿足指數函數的運算方式與性質，故

E(x)是一種指數函數。故想將它用指數的方式表出。但其底爲何？

By②，其底數=e，故將E(x)改寫e^x，且稱其爲natural exponential function.

Prop:

① e^x=ln^{-1}x

② lne^x=x, ∀ x∈$\mathbb{R}$

③ e^{lnx}=x, ∀ x∈$\mathbb{R}^+$

Thm:$(e^x)'$= e^x 只有這個函數微分是自己，積分是自己。

Next proof it.